Finite-dimensional reductions of the discrete Toda chain

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Abstract

The problem of construction of integrable boundary conditions for the discrete Toda chain is considered. The restricted chains for properly chosen closure conditions are reduced to the well known discrete Painlevé equations dP_{III}, dP_V, dP_{VI}. Lax representations for these discrete Painlevé equations are found.

1 Introduction

It is well known that the Toda lattice equation

$$q_{n,xx} = e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}} \tag{1}$$

admits several different integrable discretizations [1]-[4]. Let us consider one of them [3]

$$q_{m+1,n} - 2q_{m,n} + q_{m-1,n} = \ln \frac{e^{q_{m,n+1}-q_{m,n}} + 1}{e^{q_{m,n}-q_{m,n-1}} + 1}$$

which can be also presented in variables $u_{m,n} = e^{q_{m,n}}$

$$u_{m+1,n} = \frac{u_{m,n}^2(1 + u_{m,n+1}/u_{m,n})}{u_{m-1,n}(1 + u_{m,n}/u_{m,n-1})}. \tag{2}$$

The discrete Toda chain (2) is referred also as 2-dimensional reduction of Hirota's bilinear equation [5], [6], which has applications in statistical physics and quantum field theory [7], [8].

One of the most effective methods for constructing solutions of a discrete chain is to find its integrable finite-dimensional reductions. In most cases, this pertains to its periodic closure. But there are other possibilities for truncating the chains while conserving the integrability property [9], [3], [10]. For chains that admit zero curvature representation, there is a simple and effective method for seeking cut-off constraints (boundary conditions) compatible with the conservation laws of the chain [11]-[13].

The discrete Toda chain (2) is equivalent to the matrix equation

$$L_{m+1,n}(\lambda)A_{m,n}(\lambda) = A_{m,n+1}(\lambda)L_{m,n}(\lambda), \tag{3}$$

which is a consistency condition (the zero curvature equation) of two linear equations

$$Y_{m,n+1}(\lambda) = L_{m,n}(\lambda)Y_{m,n}(\lambda), \tag{4}$$
$$Y_{m+1,n}(\lambda) = A_{m,n}(\lambda)Y_{m,n}(\lambda), \tag{5}$$

where λ is a parameter and $L_{m,n}$, $A_{m,n}$ are matrices of the following form [3]

$$L_{m,n} = \begin{pmatrix} \lambda + \frac{u_{m,n}}{u_{m-1,n}} & u_{m,n} \\ \lambda\frac{1}{u_{m-1,n}} & 0 \end{pmatrix}, \quad A_{m,n} = \begin{pmatrix} \lambda & u_{m,n} \\ \lambda\frac{1}{u_{m,n-1}} & -1 \end{pmatrix}.$$

Definition. *We will call a boundary condition*

$$u_{m,0} = F(m, u_{m,1}, u_{m-1,1}, ..., u_{m,M}, u_{m-1,M}) \tag{6}$$

compatible with zero curvature equation (3) if equation (5) at the spatial point $n = 1$

$$Y_{m+1,1}(\lambda) = A_{m,1}(\lambda)|_{u_{m,0}=F}Y_{m,1}(\lambda) \tag{7}$$

has an additional point symmetry of the form

$$\tilde{Y}_{m,1}(\tilde{\lambda}) = H(m, [u], \lambda)Y_{m,1}(\lambda), \quad \tilde{\lambda} = h(\lambda). \tag{8}$$

In other words boundary condition (6) is integrable if there exists a matrix-valued function

$$H(m, [u], \lambda) = H(m, u_{m,1}, u_{m-1,1}, ..., u_{m,k}, u_{m-1,k}, \lambda)$$

together with the involution $\tilde{\lambda} = h(\lambda)$ such that for any solution $Y_{m,0}(\lambda)$ of the equation (7) the function (8) is a solution of the same equation. This means that the following equality

$$H(m+1, [u], \lambda)A_{m,0}(\lambda) = A_{m,0}(\tilde{\lambda})H(m, [u], \lambda) \tag{9}$$

is valid.

We note that equation (9) contains three unknowns (the boundary condition $F(m, [u])$, the involution $\tilde{\lambda}$, and the matrix $H(m, [u], \lambda)$) and generally speaking it has infinite set of solutions. But if we fix a set of arguments of one of the functions $H(m, [u], \lambda)$ or $F(m, [u])$ (i.e. if we fix number k or M) we obtain additional conditions that suffice to determine the desired functions. In the section 2 we represent several kinds of boundary conditions compatible with zero curvature equation (3) of the discrete Toda chain (2). Some of them was found earlier in [3] and [11].

The boundary condition (6) reduces the chain (2) to a half-line. To obtain finite-dimensional system we impose two boundary conditions

$$u_{m,0} = F_1(m, u_{m,1}, u_{m-1,1}, ..., u_{m,M}, u_{m-1,M}), \quad u_{m,N+1} = F_2(m, u_{m,1}, u_{m-1,1}, ..., u_{m,K}, u_{m-1,K}), \tag{10}$$

$1 \leq M, K \leq N$, which are assumed to be compatible with zero curvature equation (3). According our Definition above equality (9) holds at the points $n_1 = 1$ and $n_2 = N + 1$, while

the functions $H(m,[u],\lambda)$ and $\tilde{\lambda}$ at these points are equal to the matrices $H_1 = H_1(m,[u],\lambda)$, $H_2 = H_2(m,[u],\lambda)$ and involutions $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ respectively.

Let us suppose that involutions $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ coincide (i. e. $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda}$). In this case we can construct the generating function for the integrals of motion of this system by the standard way [9]

$$g(\lambda) = trace\left(P(m,\lambda)H_1^{-1}(m,\lambda)P^{-1}(m,\tilde{\lambda})H_2(m,\lambda)\right), \tag{11}$$

where $P(m,\lambda) = L_{m,N}(\lambda)\dots L_{m,1}(\lambda)$. Similar to the continuous case [14] one can solve this system by utilizing a definite number of symmetries in addition to integrals of motion (see [13]). The set of symmetries needed can be found by using the properly chosen master symmetries.

The case $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$ is considered in the section 3. It is shown that if $N = 1$ and $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$ then the restricted chains for certain choices closure conditions are reduced to the well known discrete Painlevé equations dP_{III}, dP_V, dP_{VI} (see (59), (61), (62) below).

2 Boundary conditions consistent with zero curvature equation.

In this section we consider boundary condition of the form (6) for the discrete Toda chain (2) assuming that it's compatible with zero curvature equation. Let us start with the matrix-equation (9) which is the main equation when defining boundary conditions. It gives rise to a system of four scalar equations on elements of matrices $H(m,\lambda)$ and $H(m+1,\lambda)$ (we denote $h_{ij} = (H(m,\lambda))_{ij}$ and $\bar{h}_{ij} = (H(m+1,\lambda))_{ij}$)

$$\lambda\bar{h}_{11} + \lambda\bar{h}_{12}F = \tilde{\lambda}h_{11} + h_{21}u_{m,1}, \tag{12}$$

$$\bar{h}_{11}u_{m,1} - \bar{h}_{12} = \tilde{\lambda}h_{12} + h_{22}u_{m,1}, \tag{13}$$

$$\lambda\bar{h}_{21} + \lambda\bar{h}_{22}F = \tilde{\lambda}h_{11}F - h_{21}, \tag{14}$$

$$\bar{h}_{21}u_{m,1} - \bar{h}_{22} = \tilde{\lambda}h_{12}F - h_{22}. \tag{15}$$

Proposition 1. *Suppose that the boundary condition (6) for the discrete Toda chain (2) is compatible with zero curvature equation (3) and the corresponding matrix $H = H(m,\lambda)$ depends on temporal variable m and λ only, i. e. it doesn't depend on the dynamical variables. Then it reads as*

$$F = \frac{1}{u_{m,0}} = \alpha\mu^{-2m}u_{m,1} + \beta\mu^{-m}. \tag{16}$$

Here and below α, β, μ are arbitrary constants.

Remark. We note that boundary condition (16) was previously found in the particular case when $\mu = 1$ and $\alpha_1 = 0$, $\beta_1 = 0$, $\alpha_2 = 2$, $\beta_2 = 0$ and $\alpha_3 = 0$, $\beta_3 = 1$ (see [3]). Yu.B. Suris elaborated an algebraic structure of finite-dimensional reductions of the discrete Toda chain (2) obtained by imposing one of this boundary conditions. In the case $u_{m,0} = \infty$, $u_{m,N+1} = -\infty$ complete integrability of corresponding system are proved.

The case $\mu = 1$ and with arbitrary constants α, β has been investigated in [13]. It was shown that the corresponding finite-dimensional systems are integrated in quadratures.

Proof of Prop.1. Since the elements of the matrix H don't depend on dynamical variables it follows from the equation (13) that $h_{12} = (-\tilde{\lambda})^m a$ where $a = const$ and

$$\bar{h}_{11} = h_{22}. \tag{17}$$

If we assume that $h_{12} \neq 0$ then the boundary condition F is easily found from (15)

$$F = \frac{\bar{h}_{21} u_{m,1} - \bar{h}_{22} + h_{22}}{\tilde{\lambda} h_{12}}.$$

Substitution of expressions for h_{12} and F into (12) yields

$$-\lambda \bar{h}_{21} u_{m,1} + \lambda \bar{h}_{22} = \tilde{\lambda} h_{11} + h_{21} u_{m,1}.$$

In virtue of independence of the matrix H on dynamical variables the last equation gives $h_{21} = \left(-\frac{1}{\lambda}\right)^m b$, where $b = const$, and

$$\bar{h}_{22} = \frac{\tilde{\lambda}}{\lambda} h_{11}. \tag{18}$$

Taking into account expressions (17), (18) and independence of the function F upon the parameter λ we immediately find the boundary condition (16) where notations $\alpha = -b$ and $\mu^2 = 1/a$ are used. The matrix H and involution $\tilde{\lambda}$ take the form

$$H(m,\lambda) = \begin{pmatrix} (-1/\lambda)^{m-1} \frac{1}{\lambda+\mu} \beta \mu^{m-1} & (-1/\lambda)^m \mu^{2(m-1)} \\ -(-1/\lambda)^m \alpha & (-1/\lambda)^m \frac{1}{\lambda+\mu} \beta \mu^m \end{pmatrix}, \quad \tilde{\lambda} = \mu^2/\lambda. \tag{19}$$

Proposition is proved.

Remark. If $F = 0$, i. e. $\alpha = \beta = 0$, then we have

$$H(m,\lambda) = \begin{pmatrix} 0 & (-\lambda)^m \mu^{2(m-1)} \\ 0 & 0 \end{pmatrix}.$$

In this case system (12)-(15) has one more solution

$$H(m,\lambda) = \begin{pmatrix} b & (-\lambda)^m a \\ 0 & b \end{pmatrix}, \quad \tilde{\lambda} = \lambda.$$

Proposition 2. *Suppose that the boundary condition (6) for the discrete Toda chain (2) compatible with zero curvature equation (3) and the corresponding matrix* $H = H(m, u_{m,1}, u_{m-1,1}, \lambda)$ *depends on dynamical variables* $u_{m,1}$ *and* $u_{m-1,1}$*. Then it reads as*

$$1) \quad F = \frac{1}{u_{m,0}} = \mu^{-2m} \frac{u_{m,1} u_{m,2}}{u_{m-1,1}} + \frac{(\mu u_{m-1,1} - u_{m,1})^2}{u_{m-1,1}(\mu^{2m} - \mu^2 u_{m,1} u_{m-1,1})} + \\ + \frac{\alpha u_{m,1} + \beta(\mu^{1-m} u_{m,1}^2 + \mu^m)}{\mu^{2m} - \mu^2 u_{m,1} u_{m-1,1}}, \tag{20}$$

$$2) \quad F = \frac{1}{u_{m,0}} = \frac{u_{m,1} + u_{m,2}}{\alpha u_{m-1,1}^2} - \frac{1}{u_{m,1}}. \tag{21}$$

Remark. Consider the discrete Toda chain (2) with boundary condition of the form (16) where $\alpha = \beta = 0$ at the left endpoint and with (20) where $\mu = 1$, $\alpha = \beta = 0$ at the right endpoint

$$e^{-q_{m,0}} = 0, \tag{22}$$

$$q_{m+1,n} - 2q_{m,n} + q_{m-1,n} = \ln \frac{e^{q_{m,n+1}-q_{m,n}} + 1}{e^{q_{m,n}-q_{m,n-1}} + 1}, \quad n = 1, ..., N-1, \tag{23}$$

$$q_{m+1,N} - 2q_{m,N} + q_{m-1,N} = \ln \frac{e^{q_{m-1,N}-2q_{m,N}-q_{m,N-1}} + \frac{(e^{q_{m-1,N}} - e^{q_{m,N}})^2}{e^{2q_{m,N}}(e^{q_{m-1,N}+q_{m,N}}-1)} + 1}{e^{q_{m,n}-q_{m,n-1}} + 1}. \tag{24}$$

This system in continuous limit corresponds to the generalized Toda chain

$$e^{-q_0} = 0, \tag{25}$$

$$q_{n,xx} = e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}}, \quad n = 1, ..N-1, \tag{26}$$

$$e^{q_{N+1}} = e^{-q_{N-1}} + \frac{q_{N,x}^2}{2shq_N}, \tag{27}$$

which is related to the Lie algebras of series D_n [10]. Years ago in [3] the following discrete analogue of (25)-(27) has been suggested

$$e^{-q_{m,0}} = 0, \tag{28}$$

$$q_{m+1,n} - 2q_{m,n} + q_{m-1,n} = \ln \frac{e^{q_{m,n+1}-q_{m,n}} + 1}{e^{q_{m,n}-q_{m,n-1}} + 1}, \quad n = 1, ..., N-2, \tag{29}$$

$$q_{m+1,N-1} - 2q_{m,N-1} + q_{m-1,N-1} = \ln \frac{(e^{q_{m,N}-q_{m,N-1}} + 1)(e^{-q_{m,N}-q_{m,N-1}} + 1)}{e^{q_{m,N-1}-q_{m,N-2}} + 1}, \tag{30}$$

$$q_{m+1,N} - 2q_{m,N} + q_{m-1,N} = \ln \frac{e^{-q_{m,N}-q_{m,N-1}} + 1}{e^{q_{m,N}-q_{m,N-1}} + 1}. \tag{31}$$

Unfortunately we failed to find relation between these two discrete analogues.

Proof of Prop 2. Consider the system of equations (12)-(15). Assume that $h_{12} \neq 0$ then it follows from (15) that

$$F = \frac{\bar{h}_{21} u_{m,1} - \bar{h}_{22} + h_{22}}{\tilde{\lambda} h_{12}}. \tag{32}$$

Let us differentiate equation (13) with respect to the variable $u_{m,2}$. This leads to the equation

$$\frac{\partial(\bar{h}_{11} u_{m,1} - \bar{h}_{12})}{\partial u_{m+1,1}} \frac{\partial u_{m+1,1}}{\partial u_{m,2}} = 0. \tag{33}$$

By setting $\frac{\partial u_{m+1,1}}{\partial u_{m,2}} \neq 0$ (we can't find F in the opposite case) and integrating (33) one finds that $\bar{h}_{11} u_{m,1} - \bar{h}_{12} = g_1(u_{m,1})$ or $h_{11} = \frac{g_1(u_{m-1,1}) + h_{12}}{u_{m-1,1}}$. Analysis of the left-hand side of the equation (13) leads us to expression $h_{22} = \frac{g_1(u_{m,1}) - \tilde{\lambda} h_{12}}{u_{m,1}}$. Here and below we use $g_i(u_{m,1})$ to denote some functions depending on dynamical variables.

Substitute obtained expressions into (12)

$$\lambda \bar{h}_{12} \bar{h}_{21} u_{m,1} - \lambda \bar{h}_{12} \frac{g_1(u_{m+1,1})}{u_{m+1,1}} + \lambda \tilde{\lambda} \bar{h}_{12}^2 \frac{1}{u_{m+1,1}} + \lambda \bar{h}_{12} \frac{g_1(u_{m,1})}{u_{m,1}} =$$

$$= \tilde{\lambda}^2 h_{12} \frac{g_1(u_{m-1,1})}{u_{m-1,1}} + \tilde{\lambda}^2 h_{12}^2 \frac{1}{u_{m-1,1}} + \tilde{\lambda} h_{12} h_{21} u_{m,1} - \lambda \tilde{\lambda} h_{12} \frac{g_1(u_{m,1})}{u_{m,1}}.$$

Remind that $\frac{\partial u_{m+1,1}}{\partial u_{m,2}} \neq 0$, so we can separate the last expression on two equalities

$$\lambda\bar{h}_{12}\bar{h}_{21}u_{m,1} - \lambda\bar{h}_{12}\frac{g_1(u_{m+1,1})}{u_{m+1,1}} + \lambda\tilde{\lambda}\bar{h}_{12}^2\frac{1}{u_{m+1,1}} + \lambda\bar{h}_{12}\frac{g_1(u_{m,1})}{u_{m,1}} = g_2(u_{m,1}), \tag{34}$$

$$\tilde{\lambda}^2 h_{12}\frac{g_1(u_{m-1,1})}{u_{m-1,1}} + \tilde{\lambda}^2 h_{12}^2\frac{1}{u_{m-1,1}} + \tilde{\lambda}h_{12}h_{21}u_{m,1} - \lambda\tilde{\lambda}h_{12}\frac{g_1(u_{m,1})}{u_{m,1}} = g_2(u_{m,1}). \tag{35}$$

Let us find h_{21} from (35)

$$h_{21} = \frac{g_2(u_{m,1})}{\tilde{\lambda}h_{12}u_{m,1}} - \frac{\tilde{\lambda}g_1(u_{m-1,1})}{u_{m,1}u_{m-1,1}} - \frac{\tilde{\lambda}h_{12}}{u_{m,1}u_{m-1,1}} + \frac{\lambda g_1(u_{m,1})}{u_{m,1}^2}.$$

After that the equation (34) takes the form

$$\begin{aligned}&\bar{h}_{12}\left(\lambda\frac{g_1(u_{m,1})}{u_{m,1}} - \lambda\tilde{\lambda}\frac{g_1(u_{m,1})}{u_{m+1,1}} + \lambda^2\frac{u_{m,1}g_1(u_{m+1,1})}{u_{m+1,1}^2} - \lambda\frac{g_1(u_{m+1,1})}{u_{m+1,1}}\right) = \\ &= g_2(u_{m,1}) - \frac{\lambda}{\tilde{\lambda}}\frac{g_2(u_{m+1,1})u_{m,1}}{u_{m+1,1}}.\end{aligned} \tag{36}$$

One can easily see that the equation (14) is rewritten by means of (36) as follows

$$\left(\frac{g_1(u_{m+1,1})}{u_{m+1,1}\bar{h}_{12}} - \frac{\tilde{\lambda}}{\lambda}\frac{g_1(u_{m-1,1})}{\bar{h}_{12}u_{m-1,1}} + \tilde{\lambda}\frac{h_{12}}{u_{m,1}\bar{h}_{12}} - \tilde{\lambda}\frac{1}{u_{m+1,1}} - \frac{\tilde{\lambda}}{\lambda}\frac{h_{12}}{\bar{h}_{12}u_{m-1,1}} + \frac{1}{u_{m,1}}\right) = 0.$$

The condition $\frac{\partial u_{m+1,1}}{\partial u_{m,2}} \neq 0$ allows one to obtain the following two equalities from the last equation

$$\frac{g_1(u_{m+1,1})}{u_{m+1,1}} + \bar{h}_{12}\frac{u_{m+1,1} - \tilde{\lambda}u_{m,1}}{u_{m+1,1}u_{m,1}} = g_3(u_{m,1}), \tag{37}$$

$$\frac{\tilde{\lambda}}{\lambda}\frac{g_1(u_{m-1,1})}{u_{m-1,1}} + h_{12}\frac{\tilde{\lambda}}{\lambda}\frac{u_{m,1} - \lambda u_{m-1,1}}{u_{m,1}u_{m-1,1}} = g_3(u_{m,1}). \tag{38}$$

We can find unknown h_{12} from (37)

$$h_{12} = \frac{u_{m,1}u_{m-1,1}}{u_{m,1} - \tilde{\lambda}u_{m-1,1}}\left(g_3(u_{m-1,1}) - \frac{g_1(u_{m,1})}{u_{m,1}}\right). \tag{39}$$

Substitution of this expression for h_{12} into (38) leads to equality

$$\begin{aligned}&\frac{\lambda}{\tilde{\lambda}}g_3(u_{m,1})u_{m,1} + g_1(u_{m,1}) + \tilde{\lambda}g_1(u_{m-1,1}) + \lambda g_3(u_{m-1,1})u_{m-1,1} = \\ &= \lambda u_{m-1,1}g_3(u_{m,1}) + u_{m,1}\frac{g_1(u_{m-1,1})}{u_{m-1,1}} + g_3(u_{m-1,1})u_{m,1} + \lambda u_{m-1,1}\frac{g_1(u_{m,1})}{u_{m,1}}.\end{aligned} \tag{40}$$

Differentiation of (40) with respect to the variables $u_{m,1}$ and $u_{m-1,1}$ gives equation

$$\frac{\partial\left(\frac{g_1(u_{m-1,1})}{u_{m-1,1}} + g_3(u_{m-1,1})\right)}{\partial u_{m-1,1}} = -\lambda\frac{\partial\left(\frac{g_1(u_{m,1})}{u_{m,1}} + g_3(u_{m,1})\right)}{\partial u_{m,1}},$$

from which it follows that

$$g_3(u_{m,1}) = \left(-\frac{1}{\lambda}\right)^{m+1}c_0u_{m,1} - \frac{g_1(u_{m,1})}{u_{m,1}} + c_1(m).$$

Let $c_i = c_i(\lambda)$ and $c_i(m) = c_i(m,\lambda)$ be some functions depending only on λ and λ, m respectively.

Substitute the expression for $g_3(u_{m,1})$ into (40)

$$\frac{1}{\tilde\lambda}\left(-\frac{1}{\lambda}\right)^m c_0u_{m,1}^2 + \frac{\lambda}{\tilde\lambda}g_1(u_{m,1}) - \frac{\lambda}{\tilde\lambda}c_1(m)u_{m,1} - g_1(u_{m,1}) + c_1(m-1)u_{m,1} =$$
$$= -\left(-\frac{1}{\lambda}\right)^{m-1} c_0u_{m-1,1}^2 - \lambda g_1(u_{m-1,1}) + \lambda c_1(m-1)u_{m-1,1} - \tilde\lambda g_1(u_{m-1,1}) + \lambda c_1(m)u_{m-1,1}.$$

Left and right-hand sides of the last equality depends only on $u_{m,1}$ and $u_{m-1,1}$ respectively. Consequently $c_1(m+1) = \frac{\tilde\lambda}{\lambda}c_1(m-1)$ and

$$g_1(u_{m,1}) = \frac{1}{\tilde\lambda - \lambda}\left((-\tilde\lambda)^{m+1}c_2 + \left(-\frac{1}{\lambda}\right)^m c_0u_{m,1}^2 - \lambda(c_1(m) - c_1(m+1))u_{m,1}\right). \tag{41}$$

Return to the equality (36). Taking into consideration (39) and (41) one gets

$$\frac{g_2(u_{m,1})}{u_{m,1}} + \left(-\frac{1}{\lambda}\right)^m c_0g_1(u_{m,1}) + \lambda\frac{g_1^2(u_{m,1})}{u_{m,1}^2} - \lambda c_1(m)\frac{g_1(u_{m,1})}{u_{m,1}} =$$
$$= \frac{\lambda}{\tilde\lambda}\frac{g_2(u_{m+1,1})}{u_{m+1,1}} + \left(-\frac{1}{\lambda}\right)^{m+1}\frac{\lambda}{\tilde\lambda}c_0g_1(u_{m+1,1}) + \frac{\lambda^2}{\tilde\lambda}\frac{g_1^2(u_{m+1,1})}{u_{m+1,1}^2} - \frac{\lambda^2}{\tilde\lambda}c_1(m+1)\frac{g_1(u_{m+1,1})}{u_{m+1,1}}.$$

Analysis of the last equation shows that

$$g_2(u_{m,1}) = \left(\frac{\tilde\lambda}{\lambda}\right)^m c_3u_{m,1} - \left(-\frac{1}{\lambda}\right)^m c_0g_1(u_{m,1})u_{m,1} - \lambda\frac{g_1^2(u_{m,1})}{u_{m,1}} + \lambda c_1(m)g_1(u_{m,1}).$$

As the function F doesn't depend on parameter λ we have $\tilde\lambda = \frac{\mu^2}{\lambda}$ and

$$F = \frac{g_2(u_{m,1})}{\mu^2\bar h_{12}h_{12}} - \frac{1}{u_{m,1}}, \tag{42}$$

where

$$h_{12} = \frac{\sqrt{\lambda}}{(\lambda^2-\mu^2)}\left(-\frac{1}{\lambda}\right)^m(a_0u_{m,1}u_{m-1,1} + \mu^{2m}a_2),$$
$$g_2(u_{m,1}) = \frac{1}{(\lambda^2-\mu^2)^2}\left(\frac{1}{\lambda}\right)^{2m}(\mu^{2m}a_3u_{m,1} - \mu^2a_0^2u_{m,1}^3 + \mu^{m+1}a_0a_1u_{m,1}^2 -$$
$$-\mu^{4m+4}a_2^2\frac{1}{u_{m,1}} - \mu^{3m+2}a_2a_1),$$
$$g_1(u_{m,1}) = \frac{\sqrt{\lambda}}{(\lambda^2-\mu^2)}\left(-\frac{1}{\lambda}\right)^m\left(\frac{\mu^{2m+2}a_2}{\lambda} - a_0u_{m,1}^2 + \frac{\mu^m a_1}{\lambda-\mu}u_{m,1}\right),$$

and a_i are arbitrary constants.

The function $\bar h_{12}$ being contained in the expression for F depends on variable $u_{m+1,1}$ which is not dynamical, i. e. it can be expressed through variables $u_{m,1}$, $u_{m-1,1}$ and function F

$$u_{m+1,1} = \frac{(u_{m,1}+u_{m,1})u_{m,1}}{u_{m-1,1}(1+u_{m,1}F)}.$$

Therefore if we denote $a_0 = -a_2\mu^2$, $a_3 = a_2\mu^4(2a_2\mu+\alpha)$, $a_1 = \beta a_2\mu^2$ then the equality (42) gives boundary condition (20). The matrix H and involution $\tilde{\lambda}$ are following

$$H = \begin{pmatrix} \frac{g_1(u_{m-1,1})+h_{12}}{u_{m-1,1}} & h_{12} \\ \frac{\lambda g_2(u_{m,1})}{\mu^2 h_{12} u_{m,1}} - \frac{\mu^2(g_1(u_{m-1,1})+h_{12})}{\lambda u_{m,1}u_{m-1,1}} + \lambda\frac{g_1(u_{m,1})}{u_{m,1}^2} & \frac{\lambda g_1(u_{m,1})-\mu^2 h_{12}}{\lambda u_{m,1}} \end{pmatrix}, \quad \tilde{\lambda} = \frac{\mu^2}{\lambda}.$$

Now suppose that $h_{12} = 0$. Then the system (12)-(15) takes the form

$$\lambda\bar{h}_{11} = \tilde{\lambda}h_{11} + h_{21}u_{m,1}, \tag{43}$$

$$\bar{h}_{11} = h_{22}, \tag{44}$$

$$\lambda\bar{h}_{21} + \lambda\bar{h}_{22}F = \tilde{\lambda}h_{11}F - h_{21}, \tag{45}$$

$$\bar{h}_{21}u_{m,1} - \bar{h}_{22} = h_{22}. \tag{46}$$

The system (43)-(46) has a nontrivial solution if we assume that $\frac{\partial u_{m+1,1}}{\partial u_{m,2}} = 0$, i. e.

$$F = g_0(u_{m,1}, u_{m-1,1})\left(1 + \frac{u_{m,2}}{u_{m,1}}\right) - \frac{1}{u_{m,1}},$$

and consequently

$$u_{m+1,1} = \frac{u_{m,1}}{u_{m-1,1}g_0(u_{m,1}, u_{m-1,1})}.$$

Taking into account (44) we get from the equations (43) and (46)

$$\bar{h}_{11} = h_{11}\frac{\tilde{\lambda}u_{m-1,1} - u_{m,1}}{\lambda u_{m-1,1} - u_{m,1}}, \quad h_{21} = h_{11}\frac{\tilde{\lambda} - \lambda}{\lambda u_{m-1,1} - u_{m,1}},$$

and so (45) takes the form

$$(\lambda - \tilde{\lambda})(1 + u_{m,1}F)(u_{m+1,1} - \lambda\tilde{\lambda}u_{m-1,1}) = 0. \tag{47}$$

It implies that $u_{m+1,1} = \lambda\tilde{\lambda}u_{m-1,1}$. The other factors in (47) don't vanish. Really, if $\lambda - \tilde{\lambda} = 0$ then H is equal to the identity matrix, which gives no involution. As for the middle factor it coincides with the factor $1 + \frac{u_{m,1}}{u_{m,0}}$ which is contained in the denominator of the chain itself. In the domain of the right hand side of the chain (2) it must be different from zero. Since $\frac{\partial u_{m+1,1}}{\partial \lambda} = 0$ we have $\tilde{\lambda} = \frac{\alpha}{\lambda}$. Thus, boundary condition F takes the form (21), the matrix H and the involution $\tilde{\lambda}$ respectively are of the form

$$H = \begin{pmatrix} g(m) & 0 \\ g(m)\frac{\alpha-\lambda^2}{\lambda(\lambda u_{m-1,1}-u_{m,1})} & g(m+1) \end{pmatrix}, \quad \tilde{\lambda} = \frac{\alpha}{\lambda},$$

where $g(m) = \prod_{k=0}^{m} \frac{\alpha u_{m-1,1}-\lambda u_{m,1}}{\lambda(\lambda u_{m-1,1}-u_{m,1})}$. The proposition is proved.

3 Discrete Painlevé equations

Consider the truncated system (2), (10) in the case of different involutions $\tilde{\lambda}_1 \neq \tilde{\lambda}_2$ at the endpoints $n = 0$ and $n = 2$. So that the endpoints are taken as close as possible, i. e. $N = 1$.

The boundary conditions F_1 and F_2 imposed at $n=0$ and $n=2$ are of one of the forms represented by (16) or (20). Denote through $H_1(\lambda,m)$, $H_2(\lambda,m)$ the matrices H corresponding to each endpoint. In the case of (16) and (20) the involutions are of the form $\tilde{\lambda}_1=\frac{\mu_1^2}{\lambda}$, $\tilde{\lambda}_2=\frac{\mu_2^2}{\lambda}$. Thus the system (2), (10) takes the form

$$\frac{1}{u_{m,0}}=F_1(m,u_{m,1},u_{m-1,1}), \tag{48}$$

$$u_{m+1,1}=\frac{u_{m,1}^2(1+u_{m,2}/u_{m,1})}{u_{m-1,1}(1+u_{m,1}/u_{m,0})}, \tag{49}$$

$$u_{m,2}=F_2(m,u_{m,1},u_{m-1,1}). \tag{50}$$

It was shown in [10] that the differential-difference Toda equation (1) admits finite dimensional reductions of the Painlevé type. The same can be done in our case of purely discrete equations.

Proposition 3. *The system (48)-(50) is equivalent to the matrix equation*

$$A_m(\delta\lambda)\,M_m(\lambda)=M_{m+1}(\lambda)A_m(\lambda), \tag{51}$$

which is the consistency condition of two linear equations

$$Y_{m+1}(\lambda)=A_m(\lambda)Y_m(\lambda), \tag{52}$$

$$Y_m(\delta\lambda)=M_m(\lambda)Y_m(\lambda), \tag{53}$$

where $M_m(\lambda)=H_1\left(\frac{\mu_2^2}{\lambda},m\right)L_m^{-1}\left(\frac{\mu_2^2}{\lambda}\right)H_2^{-1}\left(\frac{\mu_2^2}{\lambda},m\right)L_m(\lambda)$ *and* $\delta=\frac{\mu_1^2}{\mu_2^2}$.

Proof. Boundary conditions (48) and (50) are consistent with zero curvature equation (3). It follows from it that equation (5) taken at the spatial points $n=1$ and $n=2$

$$Y_{m+1,1}(\lambda)=A_{m,1}(\lambda)Y_{m,1}(\lambda), \tag{54}$$

$$Y_{m+1,2}(\lambda)=A_{m,2}(\lambda)Y_{m,2}(\lambda)$$

possesses additional linear transformations

$$Y_{m,1}\left(\frac{\mu_1^2}{\lambda}\right)=H_1(\lambda,m)Y_{m,1}(\lambda), \tag{55}$$

$$Y_{m,2}\left(\frac{\mu_2^2}{\lambda}\right)=H_2(\lambda,m)Y_{m,2}(\lambda). \tag{56}$$

As we said above the system (48)-(50) is equivalent to the consistency condition of the equation (54) with the following one

$$Y_{m,2}(\lambda)=L_{m,1}(\lambda)Y_{m,1}(\lambda). \tag{57}$$

Replacing $\lambda\to\frac{\mu_2^2}{\lambda}$ in (57) and taking to account (56) gives

$$Y_{m,1}\left(\frac{\mu_2^2}{\lambda}\right)=L_{m,1}^{-1}\left(\frac{\mu_2^2}{\lambda}\right)H_2(\lambda,m)L_{m,1}(\lambda)Y_{m,1}(\lambda).$$

Substitute the last expression into (55)

$$Y_{m,1}\left(\frac{\mu_1^2}{\lambda}\right)=H_1(\lambda,m)L_{m,1}^{-1}(\lambda)H_2^{-1}(\lambda,m)L_{m,1}\left(\frac{\mu_2^2}{\lambda}\right)Y_{m,1}\left(\frac{\mu_2^2}{\lambda}\right). \tag{58}$$

Replacing again $\lambda \longrightarrow \frac{\mu_2^2}{\lambda}$ in (58) we get the equality

$$Y_{m,1}\left(\frac{\mu_1^2}{\mu_2^2}\lambda\right) = H_1\left(\frac{\mu_2^2}{\lambda}, m\right) L_{m,1}^{-1}\left(\frac{\mu_2^2}{\lambda}\right) H_2^{-1}\left(\frac{\mu_2^2}{\lambda}, m\right) L_{m,1}(\lambda) Y_{m,1}(\lambda).$$

Omit the second subindex in $u_{m,1}$. So the equation (57) is equivalent to the equation (53). The proposition is proved.

Thus the system (48)-(50) possesses Lax pair (52), (53), which is typical for the discrete Painlevé equations. Consider several illustrative examples.

Example 1. The system (48)-(50) with boundary conditions

$$\frac{1}{u_{m,0}} = \alpha_1 u_{m,1} + \beta_1, \quad u_{m,2} = \alpha_2 \mu^{2m} \frac{1}{u_{m,1}} + \beta_2 \mu^m$$

gives rise the equation on variables $u_m = u_{m,1}$

$$u_{m+1} u_{m-1} = \frac{u_m^2 + \beta_2 \mu^m u_m + \alpha_2 \mu^{2m}}{\alpha_1 u_m^2 + \beta_1 u_m + 1}, \tag{59}$$

which is one of the forms of the third discrete Painlevé equation $d - P_{III}$ [15], [16]. Check that in the continuous limit it approaches the P_{III} equation. Return to the variables $u_m = e^{q_m}$ and take $\mu = e^{2h}$, $\alpha_1 = \bar{\alpha}_1 h^2$, $\alpha_2 = \bar{\alpha}_2 h^2$, $\beta_1 = \bar{\beta}_1 h^2$, $\beta_2 = \bar{\beta}_2 h^2$. Then the equation (59) takes the form

$$q_{m+1} - 2q_m + q_{m-1} = ln \frac{1 + h^2(\bar{\alpha}_2 e^{4mh - 2q_m} + \bar{\beta}_2 e^{2mh - q_m})}{1 + h^2(\bar{\alpha}_1 e^{2q_m} + \bar{\beta}_1 e^{q_m})}.$$

Let $h \longrightarrow 0$ in the last equation then we have

$$q_{xx} = \bar{\alpha}_2 e^{4x - 2q} + \bar{\beta}_2 e^{2x - q} - \bar{\alpha}_1 e^{2q} - \bar{\beta}_1 e^q. \tag{60}$$

Substitution $e^{q(x)} = zy(z)$, $z = e^x$ in (60) gives the third Painlevé equation [17]

$$y_{zz} = \frac{y_z^2}{y} - \frac{y_z}{z} + \frac{1}{z}(Ay^2 + B) + Cy^3 + \frac{D}{y},$$

where parameters are $A = -\bar{\beta}_1$, $B = \bar{\beta}_2$, $C = -\bar{\alpha}_1$, $D = \bar{\alpha}_2$.

By using Prop.3 we can find a matrix M for zero curvature equation (51) according to the equation (59) (we denote $m_{ij} = (M)_{ij}$)

$$\begin{aligned}
m_{12} &= \frac{1}{\varphi}(\mu^{m+1}\lambda\beta_2 - \alpha_2(\mu + \lambda)\xi u_m), \\
m_{11} &= m_{12}\left(\frac{\lambda}{u_m} + \frac{1}{u_{m-1}}\right) + \frac{\mu^m \lambda}{\varphi u_{m-1}}\left(\beta_2 \xi - \mu^m(\mu + \lambda)\frac{1}{u_m}\right), \\
m_{22} &= \frac{1}{\varphi}\left(\beta_1 \beta_2 \mu^{m+1} \frac{\lambda^2}{\mu^2 + \lambda} - \alpha_2(\mu + \lambda)\eta u_m\right), \\
m_{21} &= m_{22}\left(\frac{\lambda}{u_m} + \frac{1}{u_{m-1}}\right) + \frac{\mu^m \lambda}{\varphi u_{m-1}}\left(\beta_2 \eta - \mu^m(\mu + \lambda)\beta_1 \frac{\lambda}{u_m(\mu^2 + \lambda)}\right),
\end{aligned}$$

where

$$\varphi = \frac{\mu^{2m}}{\mu + \lambda}(\alpha_2(\mu + \lambda)^2 - \mu\lambda\beta_2^2),$$

$$\xi=\frac{\mu^2\lambda\beta_1 u_{m-1}}{\mu^2+\lambda}+\frac{\mu^2 u_{m-1}+\lambda u_m}{u_m},\quad \eta=\alpha_1\lambda u_{m-1}+\frac{\beta_1\lambda(\mu^2 u_{m-1}+\lambda u_m)}{u_m(\mu^2+\lambda)}.$$

Example 2. Impose boundary condition (16) at the point $n=1$

$$\frac{1}{u_{m,0}}=\alpha_1\mu^{-2m}u_{m,1}+\beta_1\mu^{-m},$$

and (20) where $\mu=1$ at the point $n=2$

$$u_{m,2}=\frac{u_{m-1,1}}{u_{m,1}u_{m,0}}-\frac{(u_{m-1,1}-u_{m,1})^2}{u_{m,1}(1-u_{m,1}u_{m-1,1})}+\frac{(\alpha_2(u_{m,1}^2+1)+\beta_2 u_{m,1})u_{m-1,1}}{u_{m,1}(1-u_{m,1}u_{m-1,1})}.$$

Under this constraints the Toda chain (2) is reduced to the fifth discrete Painlevé equation $d-P_V$ [18]

$$(u_{m+1}u_m-1)(u_m u_{m-1}-1)=\frac{pq(u_m-a)(u_m-1/a)(u_m-b)(u_m-1/b)}{(u_m-p)(u_m-q)},\tag{61}$$

where $p=p_0\mu^m$, $q=q_0\mu^m$ and p_0, q_0, a, b are constants such as following equalities have place

$$\begin{aligned}&p_0q_0=\alpha_2,\quad p_0+q_0=-\beta_2,\\ &a+\frac{1}{a}+b+\frac{1}{b}=\alpha_1,\quad \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)=-(3+\beta_1).\end{aligned}$$

Return to the variables $u_m=e^{q_m}$ in (61) again and take $\mu=e^{-h}$. We use the same constants α_1, α_2, β_1 and β_2 as in example 1. If $h\to 0$ then we have an equation

$$q_{xx}=\bar\alpha_1 e^{2x}(1-e^{2q})+\bar\beta_1 e^x(e^{-q}-e^q)+\frac{q_x^2}{e^{2q}-1}+\frac{\bar\alpha_2(e^q+e^{-q})+\bar\beta_2}{1-e^{2q}},$$

which gives the fifth Painlevé equation by substitution $e^{q(x)}=\frac{y(z)+1}{y(z)-1}$, $z=e^x$ [17]

$$y_{zz}=\left(\frac{1}{2y}+\frac{1}{y-1}\right)y_z^2-\frac{y_z}{z}+\frac{(y-1)^2}{z^2}\left(Ay+\frac{B}{y}\right)+C\frac{y}{z}+D\frac{y(y+1)}{y-1}.$$

Here parameters are follow $8A=-\bar\beta_2-2\bar\alpha_2$, $8B=\bar\beta_2-2\bar\alpha_2$, $C=-2\bar\beta_1$, $D=-2\bar\alpha_1$.

We will use following notation

$$\begin{aligned}&h(\lambda,\mu)=\frac{\sqrt{\lambda}}{\lambda^2-\mu^2}(\mu^{2m}-\mu^2 u_m u_{m-1}),\\ &g(u_m,\lambda,\mu,\beta)=\frac{\sqrt{\lambda}}{\lambda^2-\mu^2}\left(\frac{\mu^{2m+2}}{\lambda}+\mu^2u_m^2+\beta\mu^{m+2}\frac{u_m}{\lambda-\mu}\right),\\ &f(u_m,\lambda,\mu,\alpha,\beta)=\frac{1}{(\lambda^2-\mu^2)^2}\left(\mu^{2m+4}u_m(2\mu+\alpha)-\mu^6u_m^3-\mu^{m+5}\beta u_m^2-\frac{\mu^{4m+4}}{u_m}-\mu^{3m+4}\beta\right).\end{aligned}$$

In this example functions h, $g(u_m)$ and $f(u_m)$ correspond to following functions $h(\frac{1}{\lambda},1)$, $g(u_m,\frac{1}{\lambda},1,\beta_1)$ and $f(u_m,\frac{1}{\lambda},1,\alpha_1,\beta_1)$. Therefore elements of the matrix M take the form

$$\begin{aligned}m_{12}&=\frac{u_m}{\varphi}(\mu^{2m-2}\lambda\xi_2u_{m-1}+\eta\zeta),\\ m_{11}&=m_{12}\left(\frac{\lambda}{u_m}+\frac{1}{u_{m-1}}\right)-\frac{\lambda}{\varphi u_{m-1}}(\mu^{2m-2}\lambda h u_m u_{m-1}+\eta\xi_1\lambda u_m^2),\\ m_{22}&=\frac{u_m}{\varphi}\left(\frac{\lambda^2\mu^m\beta_1\xi_2u_{m-1}}{1+\lambda\mu}+\frac{\zeta\psi}{h}\right),\\ m_{21}&=m_{22}\left(\frac{\lambda}{u_m}+\frac{1}{u_{m-1}}\right)-\frac{\lambda}{\varphi u_{m-1}}(\lambda\mu^m\beta_1 h u_m u_{m-1}+\psi\xi_1\lambda u_m^2),\end{aligned}$$

where

$$\xi_1 = g(u_{m-1}) + h, \quad \xi_2 = g(u_m) - \lambda h,$$
$$\varphi = \lambda\xi_1 u_m g(u_m) - u_{m-1}(f(u_m)u_m + g(u_m)h), \quad \eta = \frac{\lambda\beta_1\mu^{m-1}u_{m-1}}{1+\lambda\mu} + \frac{\mu^{2m-2}(u_{m-1}+\lambda u_m)}{u_m},$$
$$\zeta = f(u_m)u_m u_{m-1} - \lambda^2 h\xi_1 u_m + g(u_m)hu_{m-1}, \quad \psi = \alpha_1\lambda u_{m-1} + \frac{\lambda\mu^m\beta_1(u_{m-1}+\lambda u_m)}{u_m(1+\lambda\mu)}.$$

Example 3. Consider the chain (2) with boundary conditions (20) where μ is arbitrary constant at the point $n=0$

$$\frac{1}{u_{m,0}} = \mu^{-2m}\frac{u_{m,1}u_{m,2}}{u_{m-1,1}} + \frac{(\mu u_{m-1,1} - u_{m,1})^2}{u_{m-1,1}(\mu^{2m} - \mu^2 u_{m,1}u_{m-1,1})} + \frac{\alpha_1(\mu^{1-m}u_{m,1}^2 + \mu^m) + \beta_1 u_{m,1}}{\mu^{2m} - \mu^2 u_{m,1}u_{m-1,1}},$$

and where $\mu = 1$ at the $n=2$

$$u_{m,2} = \frac{u_{m-1,1}}{u_{m,1}u_{m,0}} - \frac{(u_{m-1,1} - u_{m,1})^2}{u_{m,1}(1 - u_{m,1}u_{m-1,1})} + \frac{(\alpha_2(u_{m,1}^2+1) + \beta_2 u_{m,1})u_{m-1,1}}{u_{m,1}(1 - u_{m,1}u_{m-1,1})}.$$

Solving these equations with regard to the variables $u_{m,2}$ and $u_{m,0}$ and substituting them into (2) one gets an equation on variables $u_m = u_{m,1}$ which is the sixth discrete Painlevé equation $d - P_{VI}$ [19]

$$\frac{(u_{m+1}u_m - p_{m+1}p_m)(u_m u_{m-1} - p_m p_{m-1})}{(u_{m+1}u_m - 1)(u_m u_{m-1} - 1)} = \frac{(u_m - ap_m)(u_m - p_m/a)(u_m - bp_m)(u_m - p_m/b)}{(u_m - c)(u_m - 1/c)(u_m - d)(u_m - 1/d)}, \tag{62}$$

where $p = p_0\mu^m$, $p_0^2 = 1/\mu$ and a, b, c, d are constants satisfying the following conditions

$$a + \frac{1}{a} + b + \frac{1}{b} = -\frac{\alpha_1}{\mu p_0}, \quad \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right) = 4 + \frac{\beta_1}{\mu},$$
$$c + \frac{1}{c} + d + \frac{1}{d} = \alpha_2, \quad \left(c+\frac{1}{c}\right)\left(d+\frac{1}{d}\right) = -(4+\beta_2).$$

One can take $\mu = e^h$ to get the continuous limit

$$q_{xx} = \frac{e^{-q} - e^q}{(1-e^{2x})(e^{q-2x} - e^{-q})}((q_x - 1)^2 + \bar{\alpha}_1(e^{q-x} + e^{x-q}) + \bar{\beta}_1) -$$
$$-\frac{e^{2x-q} - e^q}{(1-e^{2x})(e^q - e^{-q})}(q_x^2 - \bar{\alpha}_2(e^q + e^{-q}) - \bar{\beta}_2).$$

Substitution $e^{q(x)} = \frac{y(z)+\sqrt{z}}{y(z)-\sqrt{z}}$, $e^x = \frac{1+\sqrt{z}}{1-\sqrt{z}}$ gives at once the sixth Painlevé equation [17]

$$y_{zz} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)y_z^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)y_z +$$
$$+\frac{y(y-1)(y-z)}{z^2(z-1)^2}\left(A + B\frac{z}{y^2} + C\frac{z-1}{(y-1)^2} + D\frac{z(z-1)}{(y-z)^2}\right),$$

where parameters are following $8A = \bar{\beta}_2 + 2\bar{\alpha}_2$, $8B = -\bar{\beta}_2 + 2\bar{\alpha}_2$, $8C = -\bar{\beta}_1 - 2\bar{\alpha}_1$, $8D = \bar{\beta}_1 - 2\bar{\alpha}_1 + 4$.

Elements of the matrix M for the equation (62) have the form

$$m_{12} = \frac{u_m}{\varphi}(\lambda h_1^2 \psi_2 u_{m-1} - \xi_1 \zeta),$$
$$m_{11} = m_{12}\left(\frac{\lambda}{u_m} + \frac{1}{u_{m-1}}\right) + \frac{\lambda^2 h_1 u_m}{\varphi u_{m-1}}(\xi_1 \psi_1 - h_1 h_2 u_{m-1}),$$
$$m_{22} = \frac{u_m}{\varphi}\left(\frac{\lambda h_1 u_{m-1}}{u_m}\psi_2 \xi_2 - \eta\zeta\right),$$
$$m_{21} = m_{22}\left(\frac{\lambda}{u_m} + \frac{1}{u_{m-1}}\right) + \frac{\lambda^2 h_1}{\varphi u_{m-1}}(\eta \psi_1 u_m - h_2 \xi_2 u_{m-1}),$$

where we denote

$$h_1 = h(1/\lambda, \mu), \quad h_2 = h(1/\lambda, 1), \quad g_1(u_m) = g(u_m, 1/\lambda, \mu, \beta_1), \quad g_2(u_m) = g(u_m, 1/\lambda, 1, \beta_2),$$
$$f_1(u_m) = f(u_m, 1/\lambda, \mu, \alpha_1, \beta_1), \quad f_2(u_m) = f(u_m, 1/\lambda, 1, \alpha_2, \beta_2),$$
$$\xi_1 = \lambda u_m g_1(u_{m-1}) - h_1 u_{m-1}, \quad \xi_2 = g_1(u_m) - \mu^2 \lambda h_1,$$
$$\psi_1 = g_2(u_{m-1}) + h_2, \quad \psi_2 = g_2(u_m) - \lambda h_2,$$
$$\eta = \frac{f_1(u_m)u_{m-1}}{h_1 \mu^2} - \lambda g_1(u_m) - \frac{\lambda \mu^2 \xi_1}{u_m}, \quad \zeta = f_2(u_m)u_{m-1} - \lambda^2 h_1 \psi_1 + h_1 g_2(u_m)\frac{u_{m-1}}{u_m},$$
$$\varphi = \lambda h_1 \psi_1 g_2(u_m) u - h_2 u_{m-1}(f_2(u_m)u_m + g_2(u_m)h_1).$$

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